

Randomness for semi-measures

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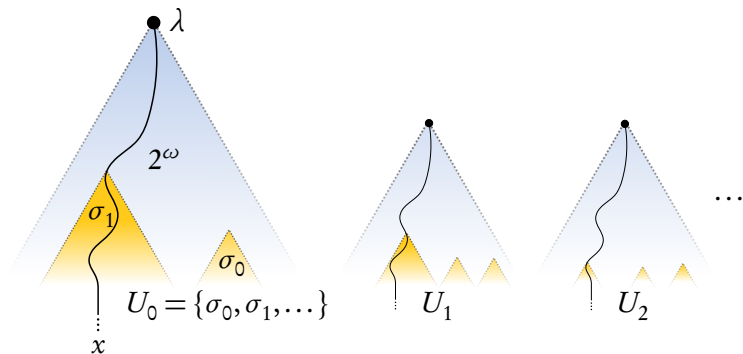
Joint work with Laurent Bienvenu, Christopher Porter and Paul Shafer

These slides (*now!*) at <http://db.tt/rYT4EcJQ>

- 1 Algorithmic randomness has been intensively studied for computable and non-computable measures.
- 2 Algorithmic randomness is closely related to computability theory; most of the work is on the interaction between both fields.
- 3 For a computability theoretic reason that we will discuss, there is a class of objects similar to measures that is relevant for algorithmic randomness, namely left-c.e. semi-measures.
- 4 We will try to understand randomness w.r.t. to these objects.

A little history

Martin-Löf randomness



- 1 Martin-Löf randomness.** Real is not random if in the intersection of a sequence of uniformly Σ_1^0 classes, whose measure tends to 0 at a guaranteed minimum speed.
- 2** Classically, the Lebesgue measure is used here.

Martin-Löf randomness for computable measures

- 1 **Definition.** A probability measure μ on 2^∞ is *computable* if $\sigma \mapsto \mu(\llbracket \sigma \rrbracket)$ is computable as a real-valued function.
- 2 **Definition.** A μ -Martin-Löf test is a sequence $(\mathcal{U}_n)_n$ of uniformly Σ_1^0 classes such that for all n , $\mu(\mathcal{U}_n) \leq 2^{-n}$.
- 3 **Definition.** $X \in 2^\infty$ is called μ -Martin-Löf random if for any μ -ML-test $(\mathcal{U}_n)_n$ we have $X \notin \bigcap_n (\mathcal{U}_n)$.

Reminder: Turing functionals

- 1 **Intuition.** A *Turing functional* effectively converts one infinite binary sequence into another.
- 2 **Definition.** A *Turing functional* $\Phi : 2^\infty \rightarrow 2^\infty$ is a (partial) function for which there exists a Turing machine M such that

$$\sigma, \sigma' \in \text{dom}(M) \wedge \sigma \sqsubseteq \sigma' \implies M(\sigma) \sqsubseteq M(\sigma')$$

For A where $M(A \upharpoonright n)$ halts for all n and $|M(A \upharpoonright n)| \rightarrow \infty$, we define $\Phi(A) = \lim_{n \rightarrow \infty} M(A \upharpoonright n)$. Otherwise $\Phi(A)$ is undefined.

- 3 **Definition.** Φ is *almost total* if $\lambda(\text{dom}(\Phi)) = 1$.

1 Let Φ be an almost total Turing functional.

2 **Definition.** The *measure induced by Φ* is

$$\lambda_{\Phi}(\sigma) = \lambda(\Phi^{-1}(\sigma)) = \lambda\{X \mid \sigma \sqsubset \Phi^X\}.$$

3 **Careful!** If Φ is *not* almost total, this need not be a measure.

4 **Proposition.** Every computable probability measure is induced by an almost total Turing functional.

5 **Theorem.** Φ almost total and $X \in \text{MLR}$ implies $\Phi(X) \in \text{MLR}_{\lambda_{\Phi}}$.

Randomness for non-computable measures

- 1 Reimann/Slaman studied random for non-computable measures.
- 2 There are two different ways of using the non-computability.
- 3 Of course we always evaluate the measure condition w.r.t. the non-computable measure.
- 4 But we have a choice of whether the procedure enumerating the test has access to the non-computable measure or not.
- 5 In the first case, we need to represent the measure somehow as an element of 2^∞ , so that the procedure can access it as oracle.
- 6 This representation will not be unique.
(as representations of real-valued functions typically are)
- 7 We will usually be interested in representations as easy as possible w.r.t. Turing reducibility.

Randomness for non-computable measures

- 1 Let μ be non-computable, and R_μ be a representation of μ .
 - An R_μ -Martin-Löf test is a sequence $(\mathcal{U}_i)_{i \in \omega}$ of uniformly $\Sigma_1^0(R_\mu)$ classes with $\mu(\mathcal{U}_i) \leq 2^{-i}$ for all i .
 - X is μ -Martin-Löf random, denoted $X \in \text{MLR}_\mu$, if there exists some R_μ for μ such that X passes all R_μ -ML-tests.
- 2 **Intuition.** μ is so “weak” that it can be represented in ways that are computationally too weak to derandomize X .

Blind randomness

- 1 Some measures are complex enough that *all* of their representations have significant derandomization power.
- 2 This interferes with randomness.
- 3 To deal with this, consider *blind* randomness, first studied by Kjos-Hanssen.
 - A *blind μ -Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i \in \omega}$ of uniformly Σ_1^0 classes with $\mu(\mathcal{U}_i) \leq 2^{-i}$ for all i .
 - X is *blind μ -Martin-Löf random*, denoted $X \in \text{bMLR}_\mu$, if X passes every blind μ -ML-test.

Left-c.e. semimeasures

- 1 A semi-measure is not guaranteed to be additive, but only to be “superadditive”.
- 2 That is, we only have $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$.
(We also allow $\rho(\emptyset) \leq 1$.)
- 3 ρ is called left-c.e. if we can uniformly in the input σ approximate $\rho(\sigma)$ from below.

- 1 We can again look at induced measures, with the same definition:

$$\lambda_{\Phi}(\sigma) = \lambda(\Phi^{-1}(\sigma)) = \lambda\{X \mid \sigma \sqsubset \Phi^X\}.$$

- 2 This time we don't require almost totality; measure loss corresponds to paths where the functional is not defined.
- 3 **Proposition (Levin/Zvonkin).** Every left-c.e. semi-measure is induced by a Turing functional.
- 4 So left-c.e. semi-measures directly correspond to Turing functionals, and are therefore natural objects to consider.
- 5 There is a universal left-c.e. semimeasure, denoted by M .

Randomness for semi-measures: the straight-forward way

- 1 Naïve definition: Plug in semi-measure instead of measure.
- 2 This notion behaves strangely.
- 3 **Proposition (BHPS).** There is a left-c.e. semi-measure ρ such that for any sequence $(\mathcal{U}_i)_{i \in \omega}$ of uniform Σ_1^0 classes we have that

$$(\forall i: \rho(\mathcal{U}_i) \leq 2^{-i}) \implies \bigcap_{i \in \mathbb{N}} \mathcal{U}_i = \emptyset.$$

- 4 In other words, all valid tests are empty.
- 5 There are no non-randoms.

What we aim for

Some (debatable) desiderata

- 1 Coherence:** If X is random with respect to μ *as measure*, we also want X to be random with respect to μ seen *as a semi-measure*.
- 2 Randomness preservation:** If $X \in \text{MLR}$ and Φ is a Turing functional, then $\Phi(X)$ is random with respect to λ_Φ .
- 3 No randomness from nothing:** If Y is random with respect to the semi-measure λ_Φ for some Turing functional Φ , then there is some $X \in \text{MLR}$ such that $\Phi(X) = Y$.

Making a measure out of a semi-measure

Repairing a semi-measure

- 1 One idea is to apply randomness definitions for measures to semi-measures.
- 2 For this we must change the semi-measure into a measure.
- 3 What differentiates a measure from a semi-measure is that the latter loses measure along the way down a path.
- 4 To fix this, decrease the measure of each parent to the sum of the measures of its two children.
- 5 This is the so-called “bar approach” by V’yugin.

Cutting back a semi-measure

- 1 V'yugin defined $\bar{\rho}(\sigma) := \inf_n \sum_{\tau \succeq \sigma \ \& \ |\tau|=n} \rho(\tau)$.
- 2 This is the largest measure such that $\bar{\rho} \leq \rho$.
- 3 For Φ inducing ρ we have $\bar{\rho}(\sigma) = \lambda(\{X : \Phi(X) \downarrow \ \& \ \Phi^X \succ \sigma\})$.
- 4 Can we use $\bar{\rho}$ to define randomness for ρ ?

$\bar{\rho}$ can be complicated

- 1 **Theorem (BHPS).** The following are equivalent for $\alpha \in (0, 1)$.
 - α is \emptyset' -right c.e..
 - There is a semi-measure ρ such that $\bar{\rho} = \alpha \cdot \lambda$.
- 2 In other words, we can make a left-c.e. semi-measure ρ such that (every representation of) $\bar{\rho}$ codes \emptyset'
- 3 **Proposition (BHPS).** There is a positive \emptyset' -computable measure μ with a low representation such that $\mu \neq \alpha \cdot \bar{\rho}$ for every left-c.e. real α and every left-c.e. semi-measure ρ .
- 4 **Open question.** Can we achieve computably dominated?

- 1 The derandomization power of \emptyset'' interferes with randomness.
- 2 So if we want to define randomness using the bar approach, we should look at the blind version, denoted by $\text{bMLR}_{\bar{\rho}}$.
- 3 **Proposition (BHPS).** There is a semi-measure ρ such that
 - $\rho = \lambda_{\Phi}$ for some Turing functional Φ ;
 - $\text{dom}(\Phi) \cap \text{MLR} \neq \emptyset$; and
 - $\text{bMLR}_{\bar{\rho}} = \emptyset$.
- 4 In other words, we have no randomness preservation.

W2R and semi-measures

Weak 2-randomness w.r.t. a semimeasure

- 1 **Definition.** For a left-c.e. semi-measure ρ , a *generalized ρ -test* is a sequence $(\mathcal{U}_i)_{i \in \omega}$ of uniformly Σ_1^0 classes with $\lim_{i \rightarrow \infty} \rho(\mathcal{U}_i) = 0$.
- 2 So this is the naïve notion of weak 2-randomness w.r.t. a semi-measure.
- 3 But it behaves well:
 - **Theorem (BHPS).** X passes every generalized ρ -test iff $X \in \text{bW2R}_{\bar{\rho}}$.
- 4 And we have preservation of randomness!
 - **Theorem (BHPS).** If $X \in \text{W2R} \cap \text{dom}(\Phi)$, then $\Phi(X) \in \text{bW2R}_{\bar{\rho}}$.
- 5 “No randomness from nothing” holds for truth-table functionals, but is open in general.

Conclusion

Intimidating diagram

$$\begin{array}{ccccccc}
 2\text{MLR}_{\rho} & \subsetneq & \text{W2R}_{\rho} & \subsetneq & \text{AR}_{\rho} & \subsetneq & \text{MLR}_{\rho} \\
 \cap & & \cap & & \cap & & \cap
 \end{array}$$

$$\begin{array}{ccccccc}
 \text{b2MLR}_{\bar{\rho}} & \subsetneq & \text{bW2R}_{\bar{\rho}} & \subsetneq & \text{bAR}_{\bar{\rho}} & \subseteq & \text{bMLR}_{\bar{\rho}} \\
 \cap & & \cap & & \cap & & \cap
 \end{array}$$

$$2\text{MLR}_{\bar{\rho}} \subsetneq \text{W2R}_{\bar{\rho}} \subsetneq \text{AR}_{\bar{\rho}} = \text{MLR}_{\bar{\rho}}$$

Open questions

- 1 **Question.** If Φ and Ψ are Turing functionals such that $\lambda_\Phi(\sigma) = \lambda_\Psi(\sigma)$ for every $\sigma \in 2^{<\omega}$, does it follow that $\Phi(W2R) = \Psi(W2R)$?
 - We know this is wrong for MLR, but holds for 2-random.
 - It also holds for W2R for truth-table functionals.
- 2 **Question.** If ρ is a left-c.e. semi-measure, does $\overline{\rho}$ have a least Turing degree representation?
- 3 **Question.** Does \overline{M} have a least Turing degree representation?

Open questions

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- 3 **Question.** Does \overline{M} have a least Turing degree representation?

Thanks for your attention.